Differential Geometry III: Riemannian geometry

6.1 (*Naturality of the exponential map*). Let $(M, g_{\mathcal{M}})$ and $(\mathcal{N}, g_{\mathcal{N}})$ be smooth Riemannian manifolds and let $\Phi : \mathcal{M} \to \mathcal{N}$ be an isometry. For any $p \in \mathcal{M}$, prove that the following diagram is commutative:

$$\begin{array}{ccc} T_p\mathcal{M} & \xrightarrow{d_p\Phi} & T_{\Phi(p)}\mathcal{N} \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{\Phi(p)}} \\ \mathcal{M} & \xrightarrow{\quad\Phi\quad} & \mathcal{N} \end{array}$$

6.2 Let $(\mathcal{M}, g)$ be a smooth *connected* Riemannian manifold.

(a) Suppose that $\Phi_1, \Phi_2 : \mathcal{M} \to \mathcal{M}$ are two isometries such that, for some $p \in \mathcal{M}$:

$$\Phi_1(p) = \Phi_2(p) \quad \text{and} \quad d_p\Phi_1 = d_p\Phi_2.$$

Prove that $\Phi_1 = \Phi_2$.

(b) Let $X \in \Gamma(\mathcal{M})$ be a Killing vector field of $(\mathcal{M}, g)$ for which there exists a point $p \in \mathcal{M}$ such that

$$X|_p = 0, \quad \nabla X|_p = 0.$$

Prove that $X = 0$.

6.3 Let $(\mathcal{M}, g)$ be a connected Riemannian manifold, and let $N \subset \mathcal{M}$ be a smooth submanifold of $\mathcal{M}$.

(a) For any $p \in \mathcal{M}$, we will define the distance of p from $\mathcal{N}$ to be

$$d(p, \mathcal{N}) = \inf \big\{ \ell(\gamma) : \gamma : [0,1] \to \mathcal{M} \text{ is a } C^1 \text{ curve, } \gamma(0) = p,\ \gamma(1) \in \mathcal{N} \big\}.$$

Assume that, for a given $p \in \mathcal{M}$, a minimizer for $d(p, \mathcal{N})$ exists, i.e. there exists a C^1 curve $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$, $\gamma(1) = q \in \mathcal{N}$ and

$$\ell(\gamma) = d(p, \mathcal{N}).$$

Show that γ is a geodesic of $(\mathcal{M}, g)$ and $\dot{\gamma}(1)$ is normal to $T_q\mathcal{N}$.

(b) Let q_1, q_2 be two points on $\mathcal{N}$ and let $\gamma : [0,1] \to \mathcal{N}$ be a C^1 curve such that $\gamma(0) = q_1$, $\gamma(1) = q_2$ and $\ell(\gamma)$ is minimal among all curves connecting q_1 to q_2 in $\mathcal{N}$, i.e.

$$\ell(\gamma) = \min \big\{ \ell(\bar{\gamma}) :\ \bar{\gamma} : [0,1] \to \mathcal{N},\ \bar{\gamma}(0) = q_1,\ \bar{\gamma}(1) = q_2 \big\}$$

Prove that, for any $t \in [0,1]$, there exists a parametrization of γ for which

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) \text{ is orthogonal to } T_{\gamma(t)}\mathcal{N} \subset T_{\gamma(t)}\mathcal{M}$$

(where ∇ is the Levi-Civita connection of $(\mathcal{M}, g)$).

6.4 (a) Let $(\mathbb{H}^2, g_{\mathbb{H}})$ be the Poincare half plane (see also Exercise 5.2): $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ and

$$g_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}.$$

Let also $\mathbb{D}^2$ be the unit disc in $\mathbb{R}^2$, equipped with the metric

$$g_{\mathbb{D}} = 4\frac{dx^2 + dy^2}{\big(1 - x^2 - y^2\big)^2}.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, show that the map $\Phi : \mathbb{D}^2 \to \mathbb{H}^2$ given by

$$\Phi(z) = -i\frac{z+1}{z-1}$$

is an isometry ($(\mathbb{D}^2, g_{\mathbb{D}})$ is known as the *Poincaré disc*; both $(\mathbb{H}^2, g_{\mathbb{H}})$ and $(\mathbb{D}^2, g_{\mathbb{D}})$ are models for the hyperbolic plane).

(*b) Let p be a point in the hyperbolic plane. Compute the metric in polar coordinates around p. *(Hint: Working in the Poincaré disc model, it suffices to only consider the case when p is at the origin, since any point $p \in \mathbb{D}^2$ can be mapped to any other point in $\mathbb{D}^2$ via an isometry. What are the geodesics in $(\mathbb{D}^2, g_{\mathbb{D}})$ emanating from the origin?)*

(*c) How is the round metric $(\mathbb{S}^2, g_{\mathbb{S}^2})$ expressed in polar coordinates around a point $p \in \mathbb{S}^2$?